SOLUTION TO PROBLEM 152[3]
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Problem 3 of Section 6.4 on p. 152. Let $\{f_k\}$ be a sequence of differentiable function on (a,b) and suppose there is a point $c \in (a,b)$ such that the series $\displaystyle\sum_{k=1}^{\infty} f_k(c)$ converges. Suppose also that the sequence of derivatives $\{f_k'\}$ satisfies $|f_k'(x)| \le M_k$ on (a,b) and that the series $\displaystyle\sum_{k=1}^{\infty} M_k$ converges. Then prove that

$$F(x) = \sum_{k=1}^{\infty} f_k(x) \qquad \text{and} \qquad G(x) = \sum_{k=1}^{\infty} f_k'(x)$$

converge on (a,b) and F is differentiable with derivative G on (a,b).

Students were not sufficiently careful to solve this subtle and difficult problem. Many failed to provide any details for the steps suggested. Here is the fleshed out solution following the outline from class. Compare your solution to this one. You can also find this written up in Rudin, *Principles of Mathematical Analysis, 3rd ed.*, McGraw-Hill, 1976, pp. 149–153. The solution uses the following Lemma, which says if one limit is uniform then the order of limits may be interchanged.

Lemma. *Let $\{h_n(x)\}$ and $h(x)$ be functions defined for $x \in (a,b)$ and $n \in \mathbb{N}$. Assume $h_n(x) \to h(x)$ converges uniformly on (a,b) as $n \to \infty$ and that $\lim_{t\to x} h_n(t) = A_n$ for each n. Then the $\{A_n\}$ converges to a number A and $\lim_{t\to x} h(t)$ exists and equals A. In other words, we may exchange the limits*

$$\lim_{t\to x}\lim_{n\to\infty} h_n(t) = \lim_{n\to\infty}\lim_{t\to x} h_n(t).$$

Proof. First we show that $\{A_n\}$ is a Cauchy sequence. Choose $\epsilon > 0$. Since the convergence $h_n(x) \to h(x)$ is uniform on (a,b) it is a uniformly Cauchy sequence. There is an $N \in \mathbf{R}$ so that

$$|h_n(t) - h_m(t)| < \epsilon \qquad \text{whenever } t \in (a,b) \text{ and } m \ge N \text{ and } n \ge N.$$

Hence

$$|A_n - A_m| = \left|\lim_{t\to x}[h_n(t) - h_m(t)]\right| = \lim_{t\to x} |h_n(t) - h_m(t)| \le \epsilon$$

whenever $m \ge N$ and $n \ge N$. Hence $\{A_n\}$ is a Cauchy sequence, thus convergent, say $A_n \to A$ as $n \to \infty$.

To show that $\lim_{t\to x} h(t) = A$, choose $\epsilon > 0$. Since $h_n(x) \to h(x)$ uniformly on (a,b), there is an $N_1 \in \mathbf{R}$ such that

$$|h(t) - h_n(t)| < \frac{\epsilon}{3} \qquad \text{for all } t \in (a,b) \text{ and any } n \ge N_1.$$

Since $A_n \to A$, there is an $N_2 \in \mathbf{R}$ so that

$$|A_n - A| < \frac{\epsilon}{3} \qquad \text{whenever } n \ge N_2.$$

Fix a number $n > \max\{N_1, N_2\}$. For this n, since $h_n(t) \to A_n$ as $t \to x$, there is a $\delta > 0$ so that

$$|h_n(t) - A_n| < \frac{\epsilon}{3}, \qquad \text{whenever } t \in (a,b) \text{ and } |t - x| < \delta.$$

Thus,

$$|h(t) - A| \le |h(t) - h_n(t)| + |h_n(t) - A_n| + |A_n - A| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever $t \in (a,b)$ and $|t - x| < \delta$. Hence $h(t) \to A$ as $t \to x$ as claimed. □

Solution of the problem. First we show that $G(x) = \sum_{k=1}^{\infty} f_k'(x)$ converges uniformly on (a,b). We are given that $|f_k'(x)| < M_k$ and $\sum_{k=1}^{\infty} M_k$ converges. By the Weierstrass M-test, it follows that $G(x)$ converges uniformly on (a,b).

Second we show that $F(x)$ converges. Fix $x \in (a,b)$. By the Mean Value Theorem, since f_k is differentiable on (a,b), there is a c_k between c and x such that

$$f_k(x) = f_k(c) + f_k'(c_k)(x - c).$$

Since $|f_k'(c_k)(x - c)| < M_k(b - a)$ and $\sum_{k=1}^{\infty} M_k(b - a)$ is summable, by the Weierstrass M-test, $\sum_{k=1}^{\infty} f_k'(c_k)(x - c)$ converges. We are given that the first term on the right is also summable. But infinite sums may be added, so $F(x)$ converges:

$$F(x) = \sum_{k=1}^{\infty} \left(f_k(c) + f_k'(c_k)(x - c)\right) = \left(\sum_{k=1}^{\infty} f_k(c)\right) + \left(\sum_{k=1}^{\infty} f_k'(c_k)(x - c)\right).$$

Third, we apply Lemma 1 to show that $F(x)$ is differentiable and $F'(x) = G(x)$. Put

$$F_n(x) = \sum_{k=1}^{n} f_k(x) \qquad \text{and} \qquad G_n(x) = \sum_{k=1}^{n} f_k'(x)$$

so $F_n(x) \to F(x)$ on (a,b) as $n \to \infty$ and $G_n(x) \to G(x)$ uniformly on (a,b) as $n \to \infty$. On the one hand if the limit existed,

$$F'(x) = \lim_{t \to x} \frac{F(t) - F(x)}{t - x} = \lim_{t \to x} \lim_{n \to \infty} \frac{F_n(t) - F_n(x)}{t - x}.$$

On the other, since we can differentiate FINITE sums termwise, $F_n'(x) = G_n(x)$. Thus

$$G(x) = \lim_{n \to \infty} F_n'(t) = \lim_{n \to \infty} \lim_{t \to x} \frac{F_n(t) - F_n(x)}{t - x}.$$

The point of the problem is to justify the exchange of limits. To apply the Lemma, put

$$h_n(t) = \begin{cases} \dfrac{F_n(t) - F_n(x)}{t - x}, & \text{if } t \ne x; \\ F_n'(x), & \text{if } t = x. \end{cases}$$

By using cases, our function $h_n(t)$ is defined for all $t \in (a,b)$. Fix $x \in (a,b)$. To apply the Lemma, we show the limit exists:

$$\lim_{t \to x} h_n(t) = \lim_{t \to x} \frac{F_n(t) - F_n(x)}{t - x} = F_n'(x) = G_n(x).$$

Thus $A_n = G_n(x)$. We also have to show that $h_n(t) \to h(t)$ uniformly on (a, b). For $t \neq x$, since f_n is differentiable, there is a c_n between t and x such that

$$\left|\frac{f_n(t) - f_n(x)}{t - x}\right| = \left|\frac{f_n'(c_n)(t - x)}{t - x}\right| = |f_n'(c_n)| \leq M_n.$$

If $t = x$ then we get the same bound $|f_n'(x)| \leq M_n$. By the Weierstrass M-test, the sum of these expressions

$$h(t) = \lim_{n\to\infty} \sum_{k=1}^{n} h_k(t) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{f_n(t) - f_n(x)}{t - x}, & \text{if } t \neq x; \\ \displaystyle\sum_{k=1}^{\infty} f_k'(x), & \text{if } t = x. \end{cases} = \begin{cases} \dfrac{F(t) - F(x)}{t - x}, & \text{if } t \neq x; \\ G(x), & \text{if } t = x \end{cases}$$

converges uniformly on (a, b). Thus we have verified the conditions of the Lemma. Its conclusion tells us that $A_n \to A$ as $n \to \infty$, in other words

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} G_n(x) = G(x)$$

which we knew already, and that the limit in the other order exists and equals A, namely,

$$G(x) = A = \lim_{t\to x} h(t) = \lim_{t\to x} \frac{F(t) - F(x)}{t - x} = F'(x)$$

as to be proved. $\square$